Reverse Mathematics, Computability and Effective Algebra

Valentin Bura

Reverse Mathematics

- A program in Mathematical Logic introduced by Harvey Friedman in 1974 at the Vancouver ICM
- The purpose is to establish the “foundations” of ordinary Mathematical Results in terms of Subsystems of Second Order Arithmetic

Computability Theory

- Originates in the work of Alan Turing, who introduced the notion of "computability" in a famous paper from 1936
- It used to be known as Recursion Theory, due to the work of Alonzo Church and Stephen Kleene
- Historically, it developed due to David Hilbert's address at the 1900 ICM in Paris
- One of the problems posed by Hilbert was the development of a "mechanical procedure" that would output all of the "truths of Mathematics"

Arithmetical Hierarchy

- We quantify over sets of Natural Numbers
- A set is Σ_1 if it is equivalent to a formula of the form

$$\exists x\ P(x)$$ where P is a computable predicate

- In this context, "computable" means Turing Machine computable
- Similarly, a set is Π_1 if it is equivalent to a formula of the form

$$\forall x\ P(x)$$ where P is a computable predicate

Sigma and Pi

Imagine you perform a search through a collection of objects, trying to find an object that satisfies a specification

Provided the collection is infinite, this question corresponds to deciding a Σ_1 property! Also known as Recursively Enumerable or Semi-Decidable

Imagine you perform a search through such a collection, trying to establish if all of the objects satisfy a specification

What is actually computed, is a Π_1 property, also known as Co-R.E.

Arithmetical Hierarchy

- This can be extended as follows
- A set is Σ_n if it is equivalent to a formula of the form
    - $\exists x\ P(x)$ where $P(x)$ is Π_{n-1}
- A set is Π_n if it is equivalent to a formula of the form
    - $\forall x\ P(x)$ where $P(x)$ is Σ_{n-1}
- A set is Δ_n if it is both Σ_n and Π_n
- In particular, a Δ_1 set is what is called decidable

Reverse Mathematics

- Five subsystems of Second Order Arithmetic are used
  - RCA_0 recursive comprehensive axiom
  - WKL_0 weak König's lemma
  - ACA_0 arithmetic comprehensive axiom
  - ATR_0 arithmetic transfinite recursion
  - Π_1^1CA pi-1-1 comprehension axiom

Typically, we work "within" one of these systems – usually RCA_0, and prove a given result of ordinary mathematics is equivalent to one of these axioms

Recursive Comprehension Axiom

- This system comprises of three main ingredients
  - Peano Axioms without induction – i.e. Robinson Arithmetic
  - Induction on Σ_1 sets
  - Comprehension for Δ_1 sets

This means results carried fully within this system must be fully effective.

Intuitively, a Turing Machine with no oracle access can produce all of the objects the proof makes use of.

A fully effective result

The rational numbers form an ordered field

This result is fully effective!

In a computable ordered field, such as the rationals

- The field operations are computable
- The inverses are computable
- The total order is computable

Weak König's Lemma

- As the name suggests, a "weak" comprehension principle
  - *Every infinite binary branching tree has an infinite path*
  - Strictly stronger than previous system – an infinite path is not computable as such, it can only be approximated!
  - That is, the infinite path property is R.E.

An equivalent result

A continuous real function on the closed unit interval <u>is bounded</u>

Boundedness on unit interval

We prove this from WKL_0, the other direction is similar

Let f(x) be a real continuous function on the closed unit interval.

Encode the function as a binary branching tree as follows
- let ∅ be the root,
- branch twice for f(0) and f(1),
- branch again twice from f(0) for f(0) and f(1/2), and
- branch twice from f(1) for f(1/2) and f(1), and so on recursively.

Now, this process approximates the function discretely.

Take the maximum of any infinite path of this tree, and that is a bound for the function.

Arithmetic Comprehension

- Two equivalent formulations
  - Proofs encoding full König’s lemma
    - *Any finitely branching infinite tree has an infinite path*
  - Proofs requiring the Halting Set as an oracle

Strictly stronger than previous system.

Intuitively, the length of the KL tree is infinite, while its width is finitely bounded – we do not know just how big the width will be, all we know is it is finite.

A result requiring arithmetic comprehension

Every bounded increasing sequence of real numbers has a limit

Convergence

We prove this from KL, the other direction is similar

Let a_i be a bounded increasing sequence of reals.
Encode a finitely branching tree from a_i as follows.

- let ∅ be the root,
- at level 1 of the tree, for all i such that $a_1 - a_i < 1$ we let a_i be a child of the root
- in general at level j of the tree, for all i such that $a_j - a_i < j$ we let a_i be a child of the previous node

This tree is infinite. It is finitely branching. By KL it contains an infinite path.
Perform a maximum operation over this path to obtain the limit of the sequence.

Our results in 2012

Reverse Mathematics of Divisibility in Integral Domains

M.Sc. Thesis, Valentin Bura,
Victoria University of Wellington

Commutative Ring Theory

- In an integral domain, if every irreducible is a prime, then every element has at most one decomposition into irreducibles

  → This result is provable in RCA_0

- Well-foundedness of divisibility implies the existence of an irreducible factorization for each element.

  → This result is equivalent to ACA_0 over the base system RCA_0

Thank you!